

State protection by quantum control before and after noise

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We discuss the possibility of protecting the state of a quantum system that goes through noise by measurements and operations before and after the noise process. We extend our previous result on nonexistence of “truly quantum” protocols that protect an unknown qubit state against the depolarizing noise better than “classical” ones [Phys. Rev. A, 95, 022321 (2017)] in two directions. First, we show that the statement is also true in any finite-dimensional Hilbert spaces, which was previously conjectured; the optimal protocol is either the do nothing protocol or the discriminate and reprepare protocol, depending on the strength of the noise. Second, in the case of a qubit, we show that essentially the same conclusion holds for any unital noise. These results describe the fundamental limitations in quantum mechanics from the viewpoint of control theory.

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I. INTRODUCTION

Quantum information technology, such as quantum computation, quantum cryptography, etc., is a new framework of information processing where quantum states (e.g. qubits) bear information in place of classical bits. One of the difficulties in realization of those technologies is existence of noise. Since there is no isolated physical system in the world, the state inevitably undergoes noise processes caused by interactions with the environment. The state evolution becomes irreversible and errors occur in information processing. In order to reduce the errors and make information processing feasible, protection of quantum states is an important task [1–5].

In the classical world, one can in principle protect any state against any noise, by taking the complete record of the state before the noise affects the system. In the quantum world, it is not the case even if the state is not a probabilistic mixture. If one could do so, then one could suppress the disturbance caused by measurements and realize disturbance-free measurements. This would contradict with quantum measurement theory [6, 7], which implies that quantum measurements cannot extract the full information from a single sample and inevitably disturb the state. Thus impossibility of perfect state protection reflects the nature of quantum mechanics, in the same way as impossibility of perfect state discrimination [8–11] or quantum cloning [12].

Given this impossibility of perfect state protection, one may still want to consider control protocols which suppress the noise approximately. This is similar to pursuing the error-disturbance uncertainty relation [13, 14] or theory of imperfect cloning [15, 16]. Quantitative analysis of the limits in state protection may reveal the role played

by measurements in state protection and whether there exists a comprehensive point of view to achieve the optimal state protection. That may also clarify the fundamental limitations in our ability to manipulate quantum systems and provide an operational characterization of the quantum world.

We would like to refer to recent works in the context of ex-ante-ex-post control scheme [17]. The scheme consists of a general measurement before the noise process (ex-ante control), and an operations after the noise process depending on the outcomes of the measurements (ex-post control), as depicted in Fig. 1. The optimal protocol which protects two states of a qubit solely by ex-post control has been derived by Branczyk *et al.* [18] and Mendonça *et al.* [19]. An interesting interpretation of their results is that the optimal protocol detects the influence of the noise without discriminating the input states at all. Whether this strategy can be extended to other situations would be an intriguing question. On the other hand, Zhang *et al.* [20] showed that the ex-post control alone cannot protect a completely unknown pure state against the depolarizing noise. The present authors, in the qubit case, extended their results to general noise [17]: the optimal ex-post control protocol to protect a completely unknown pure state against an arbitrary noise is a unitary operation, i.e., it is never beneficial to extract information in ex-post control. It is suggested by all these results that prior knowledge of the input pure states is essential to protect them. Thus, if one has no information on the input, one needs ex-ante control. Ex-ante control was considered by Korotkov and Keane [21] and then by Wang *et al.* [22]. Although their interests are the protocols with postselection, one can find in Ref. [22] some numerical results that suggest the existence of nontrivial ex-ante-ex-post control protocols (without postselection) which suppress the amplitude damping noise well. On the other hand, the present authors [17] proved that there is no nontrivial ex-ante-ex-post control that can suppress the depolarizing noise better than “classical” protocols i.e. the “do nothing”

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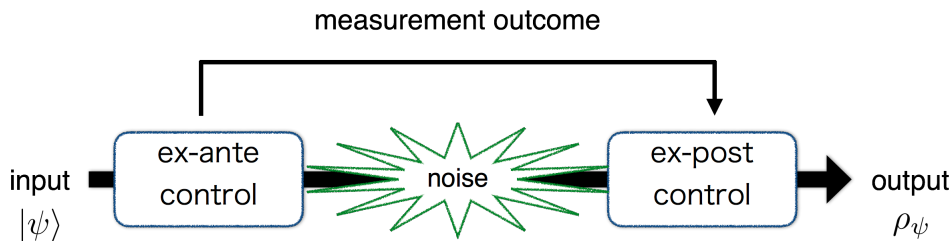


FIG. 1. The schematic diagram of the quantum control (ex-ante control and ex-post control).

and the “discriminate and reprepare” protocols, where the latter consists of an ex-ante strong measurement and an ex-post reparation of the state corresponding to the outcome of the measurement. Given the suggested existence of nontrivial quantum protocol for the amplitude damping noise and the nonexistence of such for the depolarizing noise, it is natural to ask which class of noise allows or disallows nontrivial quantum state protection.

In this paper, we extend our previous results in Ref. [17], on protection of a qubit against the depolarizing noise, in two directions. We thereby partially solve the problem of protecting a completely unknown states against noise by ex-ante-ex-post control. The first direction is to extend the results to higher dimensional Hilbert spaces. To achieve that, we provide two observations that are powerful and quite general. One is convexity of the space of noise that are optimally suppressed by the same protocol; the other is a sufficient condition on the noise for the discriminate and reprepare protocol to be optimal. As an application of these observations, we prove the conjecture in Ref. [17] that either the do nothing or the discriminate and reprepare protocol is optimal to suppress the depolarizing noise in general. The second direction is to widen the types of noise in the case of a qubit. The class of noise considered is “unbiased” or *unital* noise, which leaves the completely mixed state unchanged. The class contains many types of noise appearing in quantum information including the depolarizing noise, but does not contain the amplitude damping noise. It can be said that unital noise makes any state more random because it never decrease the (von Neumann) entropy of a quantum state. We show that the optimal ex-ante-ex-post control protocol to suppress unital noise is either a no measurement protocol or the discriminate and reprepare protocol.

The paper is organized as follows. After a very short review of basic mathematical tools in Sec. II, the state protection scheme by ex-ante-ex-post control is introduced in Sec. III. We give general observations on noise suppression and show that the “classical” protocols suffice in state protection against the depolarizing noise in Sec. IV. Then we focus on the qubit case; we review the geometry of the space of unital noise in Sec. V and show that the “classical” protocols are optimal in Sec. VI. Sec. VII is devoted to conclusion and discussions.

II. BASICS OF QUANTUM OPERATIONS

We shall introduce the basic mathematical tools and notation used in our analysis. Throughout the paper, we consider physical systems which are represented by a finite-dimensional Hilbert space. Let \mathcal{H} be such a Hilbert space and $\mathcal{L}(\mathcal{H})$ be the set of all linear operators on \mathcal{H} . A quantum state is described by a density operator $\rho \in \mathcal{L}(\mathcal{H})$ such that $\rho \geq 0$ and $\text{Tr } \rho = 1$. Since the control of quantum states consists of measurement and operations, the mathematical map corresponding to measurement or operations is explained below.

Any physical evolution of a quantum state corresponds to a trace-preserving completely positive (TPCP) map, and vice versa (e.g. [23]). Here, a linear map $\mathcal{E} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ is said *positive* if $X \geq 0$ implies $\mathcal{E}(X) \geq 0$ and *completely positive (CP)* if the map $\mathcal{E} \otimes \text{id}_n$ is positive for every positive integer n , where id_n denotes the identity map on $\mathcal{L}(\mathbb{C}^n) = \mathbb{C}^{n \times n}$. The map \mathcal{E} is said *trace-preserving (TP)* if $\text{Tr } \mathcal{E}(X) = \text{Tr } X$ for any $X \in \mathcal{L}(\mathcal{H})$.

Any physical measuring process corresponds to a CP instrument, and vice versa [7]. Here, a *CP instrument* is a family $\{\mathcal{J}_\omega\}_{\omega \in \Omega}$ of CP maps with $\sum_{\omega \in \Omega} \mathcal{J}_\omega$ being trace-preserving. We assume that the set Ω of outcomes is finite throughout the paper. The state evolution by the measurement is described as

$$\rho \mapsto \frac{\mathcal{J}_\omega(\rho)}{\text{Tr } \mathcal{J}_\omega(\rho)}, \quad \text{with probability } \text{Tr } \mathcal{J}_\omega(\rho). \quad (1)$$

The probability distribution of measurement outcomes is described by a positive operator valued measure (POVM), which is a family $\{M_\omega\}_{\omega \in \Omega}$ of positive operators on \mathcal{H} such that $\sum_{\omega \in \Omega} M_\omega$ is the identity operator. A CP instrument $\{\mathcal{J}_\omega\}_{\omega \in \Omega}$ defines a POVM $\{M_\omega\}_{\omega \in \Omega}$ by $\text{Tr } \rho M_\omega = \text{Tr } \mathcal{J}_\omega(\rho)$ or $M_\omega = \mathcal{J}_\omega^*(1)$, where an asterisk denotes the dual map. The dual map \mathcal{E}^* of $\mathcal{E} \in \mathcal{L}(\mathcal{H})$ is defined by $\text{Tr } \mathcal{E}(X)Y = \text{Tr } X\mathcal{E}^*(Y)$. We shall say that a POVM $\{M_\omega\}_{\omega \in \Omega}$ and a CP instrument $\{\mathcal{J}_\omega\}_{\omega \in \Omega}$ above are associated with each other. A POVM has all the information on the statistical properties of the measurement outcomes, while a CP instrument $\{\mathcal{J}_\omega\}$ has further information on the resulting states after the measurement.

The space $\mathcal{L}(\mathcal{H})$ of linear operators can be regarded as a Hilbert space with the Hilbert-Schmidt inner product $\langle X, Y \rangle_{\text{HS}} := \text{Tr } X^\dagger Y$. A linear map \mathcal{E} on $\mathcal{L}(\mathcal{H})$ is inter-

preted as a linear operator on the Hilbert space $\mathcal{L}(\mathcal{H})$. The trace of such \mathcal{E} is defined by

$$\text{Tr}_{\text{HS}} \mathcal{E} := \sum_i \langle V_i, \mathcal{E}(V_i) \rangle_{\text{HS}}, \quad (2)$$

where $\{V_i\}_i$ is an orthonormal basis of the Hilbert space $\mathcal{L}(\mathcal{H})$. For example, when $\dim \mathcal{H} = 2$, an orthonormal basis of $\mathcal{L}(\mathcal{H})$ is given by $\{\sigma_\mu/\sqrt{2}\}_{\mu=0}^3$ where σ_0 is the identity operator and σ_i , $1 \leq i \leq 3$, are the Pauli operators. Then the trace is written as

$$\text{Tr}_{\text{HS}} \mathcal{E} = \frac{1}{2} \sum_{\mu=0}^3 \text{Tr} [\sigma_\mu \mathcal{E}(\sigma_\mu)]. \quad (3)$$

III. THE SETUP

We shall discuss the noise suppression in the *ex-ante-ex-post quantum control scheme* below, which was proposed in Ref. [17] (see Fig. 1). The scheme consists of the following:

1. State preparation: An unknown state is prepared.
2. Ex-ante control: A measurement is performed, which is described by a CP instrument $\{\mathcal{J}_\omega\}_{\omega \in \Omega}$.
3. Noise: The state undergoes an undesired evolution, called “noise,” described by a TPCP map \mathcal{N} .
4. Ex-post control: An operation, which depends on the measurement outcome ω of the ex-ante control, is performed on the system. This is described by a family $\{\mathcal{C}_\omega\}_{\omega \in \Omega}$ of TPCP maps.

For given noise \mathcal{N} , an ex-ante-ex-post control protocol is specified by the family $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega \in \Omega}$. As we did in Ref. [17], we focus on the case that the states prepared in Step 1 are pure and is *completely unknown*, i.e., the prior probability distribution is uniform on the unit sphere in \mathcal{H} , though one can consider more general cases within the scheme above. Our problem is to find an *optimal* ex-ante-ex-post control protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega \in \Omega}$ for given noise \mathcal{N} such that the states after the measurement with outcome ω are as close to the original state $|\psi\rangle\langle\psi|$ as possible. We evaluate the closeness by *fidelity*, which is expressed by $F(\rho, |\psi\rangle\langle\psi|) := \langle\psi|\rho|\psi\rangle$ if one of the two states is pure (e.g. [23]), and the optimality is defined by the average fidelity with respect to the probability to obtain each outcome ω and with respect to that of each input state $|\psi\rangle$.

An advantage of the choice is that the resulting averaged evaluation function, the *average fidelity*

$$\bar{F} = \int_{\|\psi\|=1} d\psi \langle\psi|\mathcal{E}(|\psi\rangle\langle\psi|)|\psi\rangle, \quad (4)$$

depends on the *average operation*

$$\mathcal{E} := \sum_{\omega=1}^M \mathcal{C}_\omega \circ \mathcal{N} \circ \mathcal{J}_\omega \quad (5)$$

which is a TPCP map. We will use the formula [17]

$$\bar{F} = \frac{d + \text{Tr}_{\text{HS}} \mathcal{E}}{d(d+1)}, \quad (6)$$

where $d := \dim \mathcal{H}$ and \mathcal{E} is the average operation (5) of the protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_\omega$.

IV. RESULTS IN GENERAL SYSTEM

In this section, we consider noise in the system of a d -dimensional Hilbert space \mathcal{H} . We first give the important property which comes from convexity of the space of noise (TPCP maps) (Proposition 1). Second, we give a sufficient condition on the noise for the discriminate and reprepare protocol to be optimal (Proposition 2); this solves the difficulties in seeking optimal ex-ante-ex-post control protocol for a wide range of noise. Third, we combine these facts to solve the optimality problem for the depolarizing noise, the case $d \geq 3$ of which was a conjecture in our previous work [17]. This serves as a demonstration of the strength of Propositions 1 and 2.

Proposition 1. *The space of all noise processes \mathcal{N} that are optimally suppressed by a single ex-ante-ex-post control protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega \in \Omega}$ is convex.*

Proof. The claim is equivalent to the following: if a control protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$ optimally suppresses two noise processes $\mathcal{N}_1, \mathcal{N}_2$ then it also optimally suppresses any mixture of them, $\mathcal{N} = (1 - \alpha)\mathcal{N}_1 + \alpha\mathcal{N}_2$, $0 \leq \alpha \leq 1$. Let $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$ be such a protocol. By (5) and (6), the optimal protocol maximizes $\text{Tr}_{\text{HS}} \mathcal{E}$. For any protocol $\{(\mathcal{J}'_\omega, \mathcal{C}'_\omega)\}$, one has

$$\begin{aligned} & \text{Tr}_{\text{HS}} \mathcal{J}'_\omega \circ \mathcal{N} \circ \mathcal{C}'_\omega \\ &= (1 - \alpha) \text{Tr}_{\text{HS}} \mathcal{J}'_\omega \circ \mathcal{N}_1 \circ \mathcal{C}'_\omega + \alpha \text{Tr}_{\text{HS}} \mathcal{J}'_\omega \circ \mathcal{N}_2 \circ \mathcal{C}'_\omega \\ &\leq (1 - \alpha) \text{Tr}_{\text{HS}} \mathcal{J}_\omega \circ \mathcal{N}_1 \circ \mathcal{C}_\omega + \alpha \text{Tr}_{\text{HS}} \mathcal{J}_\omega \circ \mathcal{N}_2 \circ \mathcal{C}_\omega \\ &= \text{Tr}_{\text{HS}} \mathcal{J}_\omega \circ \mathcal{N} \circ \mathcal{C}_\omega. \end{aligned} \quad (7)$$

Thus $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$ maximizes $\text{Tr}_{\text{HS}} \mathcal{J}'_\omega \circ \mathcal{N} \circ \mathcal{C}'_\omega$ hence it optimally suppresses \mathcal{N} . \square

Let us consider operations of the form

$$\mathcal{F}(\rho) = \sum_k \rho_k \text{Tr} M_k \rho, \quad (8)$$

in which one measures the input state ρ by a POVM $\{M_k\}$ and prepares a state ρ_k according to the measurement outcome k . Such an operation is sometimes called a quantum-classical-quantum (QCQ) channel [24].

Proposition 2. *In the scheme of ex-ante-ex-post control, any QCQ noise is optimally suppressed by the discriminate and reprepare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1, \dots, d}$ defined by*

$$\mathcal{J}_\omega(\rho) = |\phi_\omega\rangle\langle\phi_\omega| \rho |\phi_\omega\rangle\langle\phi_\omega|, \quad (9)$$

$$\mathcal{C}_\omega(\rho) = |\phi_\omega\rangle\langle\phi_\omega| \text{Tr} \rho, \quad (10)$$

where $\{|\phi_\omega\rangle\}_{\omega=1,\dots,d}$ is an arbitrary orthonormal basis of \mathcal{H} . The optimal average fidelity is

$$\bar{F}_{\text{DR}} = \frac{2}{d+1}. \quad (11)$$

We give a remark before proving the proposition. The discriminate and reprepare protocol above is to discriminate the input state between certain d orthogonal states $\{|\phi_\omega\rangle\}_{\omega=1,\dots,d}$ and reprepare the discriminated state $|\phi_\omega\rangle$ after the noise. The value of the average fidelity is (11), which is independent from \mathcal{N} and from the choice of $\{|\phi_\omega\rangle\}$. Indeed, it follows from (10) that $\mathcal{C}_\omega \circ \mathcal{N} \circ \mathcal{J}_\omega = \mathcal{C}_\omega \circ \mathcal{J}_\omega$ holds for any trace-preserving \mathcal{N} . From Eq. (3), one has

$$\text{Tr}_{\text{HS}} \mathcal{C}_\omega \circ \mathcal{J}_\omega = \frac{1}{2} \text{Tr} \mathcal{J}_\omega \circ \mathcal{C}_\omega(1) = 1. \quad (12)$$

Then the average fidelity (11) is obtained by the general formula (6) for \bar{F} .

Proof of Proposition 2. First, we observe that if \mathcal{N} is a QCQ channel, so is the average operation $\mathcal{E} = \sum_\omega \mathcal{C}_\omega \circ \mathcal{N} \circ \mathcal{J}_\omega$ for any protocol $\{(\mathcal{C}_\omega, \mathcal{J}_\omega)\}$. This is so because if \mathcal{N} is written in the form (8), then one has

$$\mathcal{E}(\rho) = \sum_{\omega,k} \mathcal{C}_\omega(\rho_k) \text{Tr}[\mathcal{J}_\omega^*(M_k)\rho], \quad (13)$$

with each $\mathcal{C}_\omega(\rho_k)$ being a state and $\{\mathcal{J}_\omega^*(M_k)\}$ being a POVM. Second, it is shown in Ref. [25] that the maximum average fidelity between the input and output states for QCQ channels \mathcal{F} is given by $\bar{F} = 2/(d+1)$. Therefore, the average fidelity \bar{F} for any protocol $\{(\mathcal{C}_\omega, \mathcal{J}_\omega)\}$ does not exceed that value. On the other hand, the value can be attained by the discriminate and reprepare protocol in the theorem. \square

The proposition above partially solve the problem of state protection by ex-ante-ex-post control; if the noise turns out to be QCQ, then the discriminate and reprepare protocol is optimal. Several equivalent conditions for a map to be QCQ is known [26]. By using one of such conditions, we can prove the conjecture proposed in Ref. [17] as a corollary of Proposition 2.

Theorem 1. *The optimal ex-ante-ex-post protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1,\dots,d}$ for the depolarizing noise*

$$\mathcal{N} = (1 - \varepsilon)\rho + \varepsilon \frac{1}{d} \text{Tr} \rho, \quad (14)$$

is given as follows.

(i) *When the noise is weak, $\varepsilon \leq d/(d+1)$, the do nothing protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1}$ with*

$$(\mathcal{J}_1, \mathcal{C}_1) = (\text{id}, \text{id}) \quad (15)$$

is optimal. The optimal average fidelity is $\bar{F}_{\text{DN}} = 1 - \varepsilon(d-1)/d$.

(ii) *When the noise is strong, $\varepsilon \geq d/(d+1)$, the discriminate and reprepare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{1 \leq \omega \leq d}$ given by (9) and (10) is optimal. The optimal average fidelity is $\bar{F}_{\text{DR}} = 2/(d+1)$.*

Proof. We first show that the noise \mathcal{N} is QCQ if and only if $\varepsilon \geq d/(d+1)$. It is known [26] that a linear map \mathcal{N} is QCQ if and only if the image of the maximally entangled state $|\Psi\rangle := (1/\sqrt{d}) \sum_i |ii\rangle$ by $\text{id} \otimes \mathcal{E}$ is separable. From (14), one has

$$(\text{id} \otimes \mathcal{N})(|\Psi\rangle\langle\Psi|) = (1 - \varepsilon) |\Psi\rangle\langle\Psi| + \varepsilon \frac{1}{d}. \quad (16)$$

The right hand side above is a mixture of the maximally entangled state and the completely mixed state. The condition that such a state is separable, hence \mathcal{E} is QCQ, is $\varepsilon \geq d/(d+1)$ [27, Sec. IVB]. It then follows from Proposition 2 that the discriminate and reprepare protocol is optimal when $\varepsilon \geq d/(d+1)$. On the other hand, the do nothing protocol gives the average fidelity

$$\bar{F}_{\text{DN}} = 1 - \varepsilon + \frac{\varepsilon}{d}, \quad (17)$$

which follows from (6) and (14). When $\varepsilon = d/(d+1)$, one has $\bar{F}_{\text{DN}} = \bar{F}_{\text{DR}} = 2/(d+1)$ so that the noise \mathcal{N} is optimally suppressed both by the discriminate and reprepare and do nothing protocols. Furthermore, when $\varepsilon = 0$, the noise $\mathcal{N} = \text{id}$ is optimally suppressed by the do nothing protocol. Therefore, by Proposition 1, any noise \mathcal{N} with $0 \leq \varepsilon \leq d/(d+1)$, a convex combination of the two cases above, is optimally suppressed by the do nothing protocol, when the optimal average fidelity is given by (17). \square

V. GEOMETRY OF UNITAL NOISE

To discuss protection of the state of a qubit against unital noise in the next section, we briefly introduce the geometry of unital TPCP maps.

A linear map \mathcal{E} on operators is said *unital* if it preserves the identity operator, $\mathcal{E}(1) = 1$. The class of unital noise appears commonly in quantum information. We can interpret unital TPCP maps as “unbiased,” because it keeps the completely mixed state. An important characteristic of a unital TPCP map \mathcal{E} is that it never decreases the von Neumann entropy $S(\rho)$ of a quantum state ρ , i.e., $S(\mathcal{E}(\rho)) \geq S(\rho)$. Thus one can say that unital noise \mathcal{E} always increases (or at least keeps) the randomness of the input state ρ . We remark that one way to understand the inequality above is to apply the well-known nonincreasing property of quantum relative entropy $S(\rho||\sigma) := \text{Tr}[\rho \ln \rho - \rho \ln \sigma]$ under a TPCP map \mathcal{E} , i.e. $S(\mathcal{E}(\rho)||\mathcal{E}(\sigma)) \leq S(\rho||\sigma)$, to the state $\sigma = 1/d$. In this section, we briefly summarize the facts about the convex structure of the space of unital TPCP maps on a qubit.

We consider the set of unital TPCP maps on $\mathcal{L}(\mathcal{H})$. From the definition of unital TPCP maps, it is easy to see that the set is convex in $\mathcal{L}(\mathcal{L}(\mathcal{H}))$, is closed under composition, and contains all unitary operations. If $\dim \mathcal{H} = 2$,

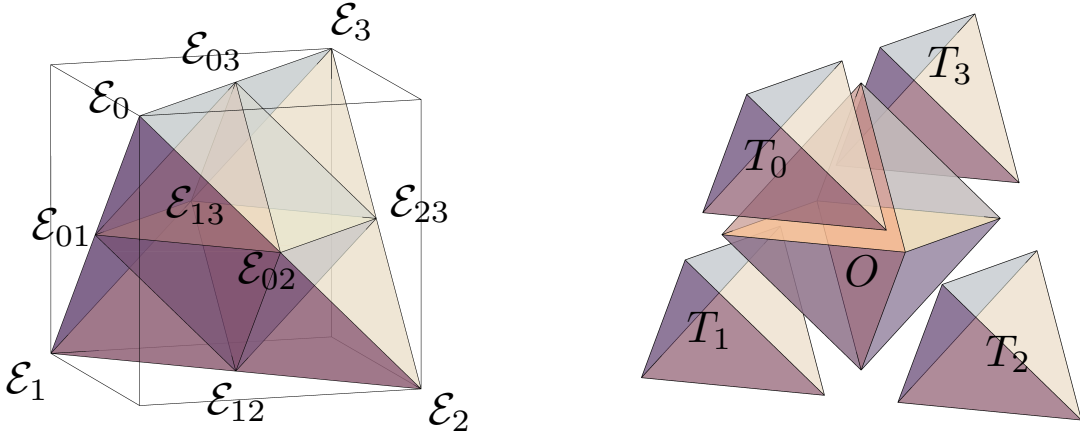


FIG. 2. Left: The space of unital noise, up to input and output unitary operations, form a tetrahedron T whose vertices are $\mathcal{E}_0(1, 1, 1)$, $\mathcal{E}_1(1, -1, -1)$, $\mathcal{E}_2(-1, 1, -1)$ and $\mathcal{E}_3(-1, -1, 1)$. Unital noise is a convex combination of the unitary operations \mathcal{E}_μ . Right: The tetrahedron T is decomposed into an octahedron O and four tetrahedra T_μ . The six vertices of O are the midpoints $\mathcal{E}_{\mu\nu}$ of the edges, which are $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$, and $(0, 0, \pm 1)$. This decomposition will be important in Theorem 2.

because each Pauli operator σ_μ is unitary, the map

$$\mathcal{N} = \sum_{\mu=0}^3 \alpha^\mu \mathcal{A}_{\sigma_\mu}, \quad \alpha^\mu \geq 0, \quad \sum_{\mu} \alpha^\mu = 1, \quad (18)$$

is unital and TPCP, where $\mathcal{A}_U(\rho) := U\rho U^\dagger$. It follows that the map

$$\mathcal{N}' = \mathcal{A}_V \circ \mathcal{N} \circ \mathcal{A}_U \quad (19)$$

is also unital and TPCP if U and V are unitary operators. Conversely, it is known [28] that the above \mathcal{N}' runs over all unital TPCP maps when we vary α^μ , U and V . Thus, apart from the degree of freedom of fixed unitary operations on the input and output states, the unital TPCP maps is parameterized by α^μ as in (18). They form a tetrahedron T with vertices at $\alpha^\mu = (1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. These vertices correspond to unitary operations \mathcal{A}_{σ_μ} .

For later calculation, we introduce a new coordinate system (d^i) by

$$\begin{bmatrix} d^1 \\ d^2 \\ d^3 \end{bmatrix} = \alpha^0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \alpha^1 \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + \alpha^2 \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} + \alpha^3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \quad (20)$$

so that

$$\mathcal{N}(\sigma_i) = d^i \sigma_i \quad (\text{no sum}) \quad (21)$$

holds. In the coordinate system (d^i) , the tetrahedron T has the vertices at $\mathcal{E}_0(1, 1, 1)$, $\mathcal{E}_1(1, -1, -1)$, $\mathcal{E}_2(-1, 1, -1)$, and $\mathcal{E}_3(-1, -1, 1)$ (Fig. 2). Let $\mathcal{E}_{\mu\nu}$ ($\mu \neq \nu$) be the midpoint of \mathcal{E}_μ and \mathcal{E}_ν , and let O be the octahedron whose vertices are the six midpoints $\mathcal{E}_{\mu\nu}$. Then the space $\overline{T} \setminus \overline{O}$ consists of four smaller tetrahedra. Let T_μ ($\mu = 0, 1, 2, 3$) be each of such tetrahedra that contains

\mathcal{E}_μ . Thus $T = O \cup T_0 \cup T_1 \cup T_2 \cup T_3$. In the following, we identify each unital noise represented by (21) [or (18)] and a point in the tetrahedron T .

We remark on the tetrahedral symmetry of T , which is the remaining symmetry on T caused by the freedom of U and V in (19). A pair (U, V) of unitary operators determines by (19) an automorphism $\mathcal{N} \mapsto \mathcal{N}'$ of the convex space of unital TPCP maps. If the pair (U, V) is properly chosen, the automorphism sends the tetrahedron T to itself so that it is a tetrahedral symmetry map. For example, when $(U, V) = (1, \sigma_3)$, the automorphism is $(d^1, d^2, d^3) \mapsto (-d^1, -d^2, d^3)$ and sends $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ to $(\mathcal{E}_3, \mathcal{E}_2, \mathcal{E}_1, \mathcal{E}_0)$. When $(U, V) = (e^{i\pi\sigma_3/4}, e^{-i\pi\sigma_3/4})$, the automorphism is $(d^1, d^2, d^3) \mapsto (d^2, d^1, d^3)$ and sends $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3)$ to $(\mathcal{E}_0, \mathcal{E}_2, \mathcal{E}_1, \mathcal{E}_3)$. Thus, the six pairs $(U, V) = (1, \sigma_i)$ and $(e^{i\pi\sigma_i/4}, e^{-i\pi\sigma_i/4})$, $1 \leq i \leq 3$, generate the tetrahedral symmetry group consisting of $4!$ maps (all permutations of the indices). In particular, four small tetrahedrons T_μ are equivalent if we disregard unitary operations before and after the noise.

VI. QUBIT UNDER UNITAL NOISE

Now we present our main result for state protection against arbitrary unital noise when $\dim \mathcal{H} = 2$. The theorem below generalize the result for the depolarizing noise [17] to general unital noise.

In general, there is a trade-off between the information gained and the disturbance caused by the ex-ante control. Though one might expect that a protocol with weak ex-ante measurements and weak ex-post control would be optimal, the theorem states that this is not the case.

Below we discuss control protocols $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$ for noise \mathcal{N} of the form (18) without loss of generality [29].

Theorem 2. *Let $\dim \mathcal{H} = 2$ and let $\mathcal{N} \in T$. Thus \mathcal{N} is unital noise of the form (21). Then the optimal ex-*

ante-ex-post control protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1,2,\dots}$ suppressing noise \mathcal{N} is given as follows.

(i) When $\mathcal{N} \in T_\mu$, the optimal protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1}$ is a no measurement protocol defined by

$$\mathcal{J}_1 = \text{id}, \quad \mathcal{C}_1 = \mathcal{A}_{\sigma_\mu}. \quad (22)$$

The optimal average fidelity is

$$\bar{F}_{\text{no measurement}} = \frac{1}{2} + \frac{|d^1| + |d^2| + |d^3|}{6}. \quad (23)$$

(ii) When $\mathcal{N} \in O$, the discriminate and repare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1,2}$, defined by (9) and (10) with $d = 2$, is optimal. The optimal average fidelity is $\bar{F}_{\text{DR}} = 2/3$.

The no measurement protocol above does not involve any measurement and merely cancels the reversible part of the noise. When $\mu = 0$, it is nothing but the do nothing protocol and the value of the average fidelity \bar{F}_{DN} can be obtained by direct calculation:

$$\begin{aligned} \bar{F}_{\text{DN}} &= \int_{\|\mathbf{x}\|=1} d\nu \text{Tr} \left[\frac{1 + \mathbf{x} \cdot \boldsymbol{\sigma}}{2} \frac{1 + \sum_i d^i x^i \sigma_i}{2} \right] \\ &= \frac{1}{2} + \frac{\sum_i d^i}{6}, \end{aligned} \quad (24)$$

where ν is the normalized uniform measure on a unit sphere. The discriminate and repare protocol appeared in Proposition 2 and $\bar{F}_{\text{DR}} = 2/3$ follows from Eq. (11) and $\dim \mathcal{H} = 2$. The no measurement and discriminate and repare protocols are considered “classical” because one either performs no quantum measurement at all or only uses the classical information extracted by the ex-ante measurement.

The difference between noise in T_μ and O can be understood as the strength of noise. In fact, the vertices \mathcal{E}_μ are unitary operations, while the origin, which always outputs the completely mixed state, entirely destroys the initial state. The theorem above states that the optimal protocol depends on the strength of the noise and suddenly changes at the threshold, with no intermediate regime in which truly quantum protocols are optimal.

Lemma 1. In the two-dimensional Hilbert space \mathcal{H} , consider the noise $\mathcal{N} = \mathcal{E}_{0i}$, $1 \leq i \leq 3$, i.e.,

$$\mathcal{N}(\rho) = \frac{1}{2}(\rho + \sigma_i \rho \sigma_i). \quad (25)$$

The do nothing and the discriminate and repare protocols are optimal ex-ante-ex-post control protocols to suppress $\mathcal{N} = \mathcal{E}_{0i}$. The optimal average fidelity is $\bar{F} = 2/3$.

Proof. We give a proof for $i = 3$; the cases $i = 1, 2$ are essentially the same. Let us show that the dephasing noise $\mathcal{N} = \mathcal{E}_{03}$ is a QCQ channel. Let P_0 and P_1 be the projection to the eigenspaces of σ_3 with eigenvalues 1 and -1 . Then, inserting $\sigma_0 = P_0 + P_1$ and $\sigma_3 = P_0 - P_1$ to

$\mathcal{N}(\rho) = \frac{1}{2}(\sigma_0 \rho \sigma_0 + \sigma_3 \rho \sigma_3)$, one can rewrite \mathcal{N} in the form (8) with $\rho_k := P_k$ and $M_k := P_k$, $k = 0, 1$. Then the optimality of the discriminate and repare protocol in suppressing \mathcal{N} follows from Proposition 2. On the other hand, the value $\bar{F} = 2/3$ can be attained also by the do nothing protocol, which can be seen by substituting $(d^i) = (0, 0, 1)$ into \bar{F}_{DN} in (24). Therefore the claim is true. \square

We give in the Appendix an alternative proof of Lemma 1 which does not depend on Proposition 2 and is based on a direct calculation. Now, let us prove Theorem 2.

Proof of Theorem 2. (i) When $\mathcal{N} \in T_0$, it is trivial that the do nothing protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1} = \{(\text{id}, \text{id})\}$ optimally suppresses the noise $\mathcal{E}_0 = \text{id}$ with $\bar{F} = 1$. From Lemma 1, this protocol also suppresses optimally the noise \mathcal{E}_{0i} , $1 \leq i \leq 3$. Then from Lemma 1, the do nothing protocol optimally suppresses any noise \mathcal{N} in the convex hull T_0 of \mathcal{E}_0 , \mathcal{E}_{01} , \mathcal{E}_{02} , and \mathcal{E}_{03} . The average fidelity \bar{F} is given by Eq. (24).

When $\mathcal{N} \in T_i$, the noise $\mathcal{A}_{\sigma_i} \circ \mathcal{N}$ is in T_0 , as explained in the preceding section, and hence optimally suppressed by the do nothing protocol $\{(\text{id}, \text{id})\}$. Therefore \mathcal{N} here is optimally suppressed by $\{(\text{id}, \mathcal{A}_{\sigma_i})\}$ [29], which is a no measurement protocol. The average fidelity \bar{F} is given by Eq. (24) with d^j ($j \neq i$) replaced with $-d^j$.

(ii) By Lemma 1, three of the vertices \mathcal{E}_{0i} of the octahedron \mathcal{O} are optimally suppressed by the discriminate and repare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$ defined by (9) and (10). The other vertices can be flipped to one of the former three by \mathcal{A}_{σ_i} , as explained in the preceding section. Thus they are optimally suppressed by the discriminate and repare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega \circ \mathcal{A}_{\sigma_i})\}$ [29]. Furthermore, the protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$ gives the same average fidelity as $\{(\mathcal{J}_\omega, \mathcal{C}_\omega \circ \mathcal{A}_{\sigma_i})\}$, because the average operation (5) yields $\mathcal{E} = \sum_\omega \mathcal{C}_\omega \circ \mathcal{J}_\omega$ for the both protocols. Thus all vertices of \mathcal{O} are optimally suppressed by the same discriminate and repare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$. Recalling that O is the convex hull of these six vertices, one concludes, by Proposition 1, that each $\mathcal{N} \in \mathcal{O}$ is optimally suppressed by the discriminate and repare protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}$. \square

VII. CONCLUSION AND DISCUSSIONS

We discussed the problem of protecting a completely unknown state against given unital noise by ex-ante and ex-post control scheme. A protocol in the scheme is described mathematically by a family of pairs, $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega \in \Omega}$, where $\{\mathcal{J}_\omega\}_{\omega \in \Omega}$ is the CP instrument with the set Ω of outcomes which describes the ex-ante measurement and the map \mathcal{C}_ω is the TPCP map which describes the ex-post operation when the outcome ω is obtained. To evaluate the closeness of the input and output states, we have chosen the average fidelity \bar{F} between the input and output states.

We presented two general observations on convexity of the noise that are optimally suppressed by the same protocol (Proposition 1) and a sufficient condition for the discriminate and reprepare protocol to be optimal (Proposition 2). These observations enabled us to prove the previous conjecture as Theorem 1, which states that the depolarizing noise is optimally suppressed by classical protocols; namely, the do nothing protocol is optimal if the noise is weak and the discriminate and reprepare protocol is optimal if the noise is strong. Then we focused on the case of a qubit system and generalized the result to the class of unital noise, which can be considered as unbiased because it preserves the completely mixed state. We proved that arbitrary unital noise is optimally suppressed by the classical protocols, the no measurement protocol or the discriminate and reprepare protocol depending on the strength of the noise (Theorem 2).

Our results suggest that one can perform nontrivial suppression of noise only by taking advantage of the bias of the noise. This gives a natural understanding for the previously known facts and numerical evidences that nontrivial suppression is possible against the amplitude damping noise but is not possible against the depolarizing noise [17, 22]. For a deeper and more precise understanding of state protection in this direction, it will be necessary to examine our Theorem 2 in higher dimensional Hilbert spaces and to investigate optimal ex-ante-ex-post control protocols against non-unital noise.

We would like to emphasize that the new method based on Propositions 1 and 2 is much more general than the previous one [17] which involved a detailed estimation of a function of several variables. Thus it may give not only a systematic approach but also a general perspective to the problem of state protection. The proofs of Theorems 1 and 2 were to find several TPCP maps which are optimally suppressed by a single protocol and to derive the optimality of the protocol in their convex hull. It was especially important to find the particular noise (TPCP map) that is optimally suppressed by two or more different protocols simultaneously, such as $\mathcal{E}_{\mu\nu}$ in the proof of Theorem 2. In other words, it is essential to find the *watersheds (critical points)* in the space of noise for determination of the *basin of optimality (convex domain)* of a protocol. Further applications of the method may reveal the “phase diagram” of optimality in the space of noise. We hope that this work serves as a prototype for such developments.

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Appendix: An elementary proof of Lemma 1

We give a proof for the case $i = 3$, the dephasing noise. The cases $i = 1, 2$ are similar. From Eq. (6), the optimal protocol $\{(\mathcal{J}_\omega, \mathcal{C}_\omega)\}_{\omega=1,2,\dots}$ is the maximizer of $\sum_\omega f_\omega$, with

$$\begin{aligned} f_\omega &:= \text{Tr}_{\text{HS}} \mathcal{C}_\omega \circ \mathcal{N} \circ \mathcal{J}_\omega = \text{Tr}_{\text{HS}} \mathcal{N} \circ \mathcal{J}_\omega \circ \mathcal{C}_\omega \\ &= \frac{1}{2} \text{Tr} \mathcal{J}_\omega \circ \mathcal{C}_\omega(1) + \frac{1}{2} \text{Tr} \sigma_z \mathcal{J}_\omega \circ \mathcal{C}_\omega(\sigma_z) \\ &= \frac{1}{2} \text{Tr} \mathcal{J}_\omega^*(1) \mathcal{C}_\omega(1) + \frac{1}{2} \text{Tr} \mathcal{J}_\omega^*(\sigma_z) \mathcal{C}_\omega(\sigma_z), \end{aligned} \quad (\text{A.1})$$

where we have used (3) and $\mathcal{N}^*(1) = 1$, $\mathcal{N}^*(\sigma_z) = \sigma_z$, $\mathcal{N}^*(\sigma_x) = \mathcal{N}^*(\sigma_y) = 0$, which follow from (25).

Let us write

$$\mathcal{C}_\omega(1) = 1 + \boldsymbol{\alpha}_\omega \cdot \boldsymbol{\sigma}, \quad \mathcal{C}_\omega(\sigma_z) = \beta_\omega \cdot \boldsymbol{\sigma}, \quad (\text{A.2})$$

$$\mathcal{J}_\omega^*(1) = \gamma_\omega + \boldsymbol{\delta}_\omega \cdot \boldsymbol{\sigma}, \quad \mathcal{J}_\omega^*(\sigma_z) = \varepsilon_\omega + \boldsymbol{\zeta}_\omega \cdot \boldsymbol{\sigma}. \quad (\text{A.3})$$

Because $\nu + \boldsymbol{\xi} \cdot \boldsymbol{\sigma} \geq 0$ holds if and only if $\|\boldsymbol{\xi}\| \leq \nu$, positivity of \mathcal{C}_ω and \mathcal{J}_ω imply $\|\boldsymbol{\alpha}_\omega \pm \boldsymbol{\beta}_\omega\| \leq 1$ and $\|\boldsymbol{\delta}_\omega \pm \boldsymbol{\zeta}_\omega\| \leq \gamma_\omega \pm \varepsilon_\omega$, respectively (Consider the images of $1 \pm \sigma_z$). One therefore has

$$\begin{aligned} f_\omega &= \gamma_\omega + \boldsymbol{\alpha}_\omega \cdot \boldsymbol{\delta}_\omega + \boldsymbol{\beta}_\omega \cdot \boldsymbol{\zeta}_\omega \\ &= \gamma_\omega + \frac{1}{2}((\boldsymbol{\alpha}_\omega + \boldsymbol{\beta}_\omega) \cdot (\boldsymbol{\delta}_\omega + \boldsymbol{\zeta}_\omega) + (\boldsymbol{\alpha}_\omega - \boldsymbol{\beta}_\omega) \cdot (\boldsymbol{\delta}_\omega - \boldsymbol{\zeta}_\omega)) \\ &\leq \gamma_\omega + \frac{1}{2}(\|\boldsymbol{\alpha}_\omega + \boldsymbol{\beta}_\omega\| \|\boldsymbol{\delta}_\omega + \boldsymbol{\zeta}_\omega\| + \|\boldsymbol{\alpha}_\omega - \boldsymbol{\beta}_\omega\| \|\boldsymbol{\delta}_\omega - \boldsymbol{\zeta}_\omega\|) \\ &\leq \gamma_\omega + \frac{1}{2}((\gamma_\omega + \varepsilon_\omega) + (\gamma_\omega - \varepsilon_\omega)) \\ &= 2\gamma_\omega, \end{aligned} \quad (\text{A.4})$$

where we have used the Cauchy-Schwarz inequality in the first inequality. It follows from trace preservation of $\sum_\omega \mathcal{J}_\omega$, or $\sum_\omega \mathcal{J}_\omega^*(1) = 1$, that $\sum_\omega \gamma_\omega = 1$ holds. As a result, one obtains $\sum_\omega f_\omega \leq 2$, which is, from Eq. (6), equivalent to

$$\bar{F} \leq \frac{2}{3}. \quad (\text{A.5})$$

This value is attained by the both of the do nothing and the discriminate and reprepare protocols, which can be seen by (24) with $(d^1, d^2, d^3) = (1, 0, 0)$ and by (11), respectively.

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